ON A POSSIBLE MANNER OF ESTABLISHING THE PLASTICITY RELATIONS

(O VOZMOZHNOM PUTI POSTROENIIA SOOTNOSHENII PLASTICHNOSTI)

PMM Vol.23, No.2, 1959, pp. 282-291

V. D. KLIUSHNIKOV
(Moscow)

(Received 31 December 1958)

All known theories of plasticity which include the concept of the angle on the yield surface contain in one form or another a specific assumption, i.e. the principle of independence of plastic response of certain elements. In the physical theories, such as those by Batdorf and Budiansky [1], Lin [2], etc., the principle of independence of plastic response is accepted for each plane or for sliding systems. In the phenomenological theories of the type of Koiter [3] and Sanders [4], the principle of independence of response is stipulated for each regular surface or plane of yielding. A survey of such theories can be found in reference [5]. However, the application of the principle of independent response in plasticity calls for certain known criticism. It can be only hoped that the errors induced into such theories are little dependent on the state of stress, and that this theory, verified in a simple experiment, gives sufficiently accurate results for arbitrary states of stress.

Therefore, it is natural to establish a theory of plasticity which includes the concept of the angular point, without application of the principle of independent response in any form. Such an attempt has been made by the author [6] for problems of plane loading paths. However, one of the assumptions introduced in that paper, namely the plasticity conditions for radial loading, is not quite satisfactory. In place of that assumption, in the present paper we consider a more general one, which seems to be the more natural assumption D. In addition, the local-minimum property of change of the plasticity curve as described in [6], which leads to angles on that curve, will be formulated somewhat differently (assumption C). The new system of assumptions leads to improved results. Thus, one of the conclusions difficult to explain on the basis of the theory advanced in [6] consisted in the fact that at the end of simple loading there existed another region besides the elastic one; if an additional loading was directed into this other region, the plastic shear modulus became equal to the elastic one. In the present paper the plastic shear modulus is dependent on the direction of loading, as in the theory by Batdorf and Budiansky, which appears to be more natural.

By virtue of the method of presentation used in [6], it may possibly be somewhat difficult to obtain an appreciation of all the assumptions which are truly necessary for the derivation of the relation "stress-strain". In the present paper we have made an attempt to delineate precisely these assumptions and to formulate them in such a manner that the following investigation bears a geometric character.

1. Definitions and restrictions. Denote by $\mathbf{P}$ the stress vector and by Э the strain vector, whose components form a deviator, and let these vectors be constructed in a nine-dimensional system of coordinates such that the deviator components with a single subscript are measured along the same axis. Further in place of the vector Э we will often investigate the vector of plastic deformations $\text{Э}^p = \text{Э} - \mathbf{P}/2G$, where GG is the elastic shear modulus. Evidently, components of this vector can be written as $\partial_{ij}{}^p = \partial_{ij} - S_{ij}/2G$. We will call the hodographs of the vectors $\mathbf{P}$, Э and Э^p the paths of stress, strain and plastic strain respectively. We call the small change of the state of stress, characterized by the vector $\Delta\mathbf{P}$, the additional loading. The magnitude of the vector $\mathbf{P}$, $\Delta\mathbf{P}$ and Э^p is denoted by p, $\Lambda\sigma$ and $'\partial^p$ respectively.

We confine ourselves to considering the case when the loading path lies in a two-dimensional plane of the nine-dimensional space indicated above. The stress-strain relation is then applicable, for example, for stability investigations. The assumed properties of the material will be formulated with respect to the plane stress vector.

2. Assumptions. A. If a relation can be established for two different paths of loading, such that the distance between two corresponding points is less than δ, then the strain between corresponding points is less than ϵ, which depends on δ and which approaches zero as ϵ approaches zero.

B. For any state $\mathbf{P}$, obtained by a given loading, there exists a closed piecewise smooth convex curve in the plane of stress, called the plasticity curve, such that moving along the curve, or inside the curve, represents purely elastic deformation of the material.

C. During the plastic deformation the plasticity curve changes continuously, possessing the following local property: during additional loading $\Delta\mathbf{P}$, the points on the plasticity curve in the neighborhood of loading may be displaced only in the direction opposite to the origin of coordinates, tending to pass along the shortest path.

D. Let there be given two states of stress $\mathbf{P}_1$ and $\mathbf{P}_2$, and the additional loading $\Lambda\mathbf{P}_1$ and $\Delta\mathbf{P}_2$ corresponding to the plastic deformation

$\Delta\text{Э}_1{}^p$ and $\Delta\text{Э}_2{}^p$ respectively. If the differential element of the curve of plasticity at the point $\mathbf{P}_1$ can be superposed by a rigid displacement or reflection, with the differential element on the curve of plasticity at point $\mathbf{P}_2$, and if the vector $\Delta\mathbf{P}_1$ can be superposed with vector $\Delta\mathbf{P}_2$, then the vectors $\Delta\text{Э}_1{}^p$ and $\Delta\text{Э}_2{}^p$ be superposed too.

E. The volumetric strain is elastic.

3. Discussion of the system of assumptions. Assumption A is the usual condition of continuity and geometrically expresses the fact that a small change in the loading process causes a small change in the deformation process. Assumption B represents a geometric interpretation of the generally accepted concept of the loading function [7].

There is reason to believe that the sequence of the plasticity curves on an arbitrary loading path is such that in the neighborhood of the points on the path the preceding plasticity curve does not pass outside the following one. This proposition is obvious in the case of a smooth plasticity curve. Its violation in the presence of angles on the plasticity curve may lead to a non-elastic response of reversed loading, which seems to us impossible [6]. It is regrettable that no special experiments have been conducted to clarify this point. The majority of experimental studies, the investigation of the laws governing the changes in the plasticity curve, have hitherto been conducted for the case of simple loading, for which the detail of the behavior of the plasticity curve indicated above is fulfilled. The most significant study in this direction is admittedly that by Naghdi, Essenburg and Koff, in which (as also in some other papers) it is pointed out that in the process of simple loading the plasticity curve changes continuously, and in some regions, near the end of the stress vector, tends to be linear, becoming more acute as the stresses increase. Assumption C is an idealization of the propositions expounded above. We note that this assumption is a special consequence of the method postulated by Sanders regarding the construction of plasticity curves.

It is known that in the process of plastic deformation the shape of the plasticity curve depends on the loading history. We may assume that this is a reversible one-to-one relation, that is a given plasticity curve corresponds to a unique loading path and vice versa. It follows from this that the vector $\Delta\text{Э}^p$ will be completely determined if the plasticity curve and the additional loading vector $\Delta\mathbf{P}$ are given. Indeed, this is true if the plasticity curve is smooth, and it further turns out that the direction of $\Delta\text{Э}^p$ is influenced not by the whole plasticity curve, but only by its differential element at the point of additional loading; thereby the influence of the differential element and the vector $\Delta\mathbf{P}$ on the direction of the vector $\Delta\text{Э}^p$ is invariant with respect to the group of rigid displacements in the stress plane, that is it is the same in the local system of coordinates connected with the differential element and the vector $\Delta\mathbf{P}$. As regards the magnitude of the vector $\Delta Э^p$, it depends on the whole plasticity curve. Since the laws of change of the plasticity curve for an arbitrary loading path are not known, the tendency has been to select a simple scalar function, whose influence on $|\Delta Э^p|$ can to a certain extent replace the influence of the plasticity curve (for example, stress intensity, plastic work, etc.).

If a singularity occurs (plasticity angle) at the point of additional loading, then together with a considerable increase of the complexity of the study, a positive characteristic appears which consists in the fact that a scalar function is introduced which is related to the point of additional loading in a natural way, namely the magnitude of the plasticity angle, which may possibly be used as a measure of plastic deformation. On the other hand, the possibility that the occurrence of an isolated singularity on the plasticity curve violates the feature that the plasticity curve influences the direction of $\Delta Э^p$ only locally, established in the case of smoothness, seems remote. At any rate, it appears that the influence of the parts of the plasticity curve remote from the point of additional loading on the direction of $\Delta Э^p$ may be neglected as compared to the influence of other factors. On the basis of what has been said above, in the case of occurrence of angles on the plasticity curve there arises the possibility (whose presence in the case of smoothness is not clarified) of determining the vector $\Delta Э^p$ completely by means of the differential element at the point of additional loading and the vector $\Delta\mathbf{P}$, that is of making this dependence a more local one. Thus, it may be assumed that the vector $\Delta Э^p$ will be completely determined if the magnitude and position of the plasticity angle and the additional loading $\Delta\mathbf{P}$ are known. Assumption D asserts that the position of the plasticity angle does not influence the relation between the vector $\Delta Э^p$ and the additional loading and represents a simple extension of the property of local invariance, which occurs in the case of smoothness for the direction of the vector $\Delta Э^p$, and for the direction and the magnitude of $\Delta Э^p$ when the plasticity curve has angles.

We note that assumption D contains more than we need for what follows. We will merely require assumption D to be satisfied when the vector $\Delta Э^p$ is along the bisectrix of the plasticity angle.

Finally, assumption E is the usual one in plasticity theory.

4. Some consequences. To determine the relationship between the vectors of stress and strain, the given loading path can be replaced by some other neighboring path, which may be composed of separate curves or of straight lines on which the plasticity laws are known or are postulated in a more natural manner than on the original path. In accordance with assumption A the deformation path found will pass over into that sought if the changed loading path approaches the given one.

From assumption B it follows that the plasticity curve changes during the process of plastic deformation in such a manner that it always passes through the tip of the stress vector. At this point the differential element of the plasticity curve, in general, is represented by an angle which we will call the plasticity angle. The magnitude of this angle will be measured by the angle γ which is formed by one of its sides with the extension of the other. A point of smoothness corresponds to the case $\gamma = 0$.

Let there be an angle γ at some point $\mathbf{P}$ of the plasticity curve. Around the point $\mathbf{P}$ we isolate a sufficiently small region H whose diameter is of the order of magnitude of future additional loading. Further, let an additional loading $\Delta\mathbf{P}$ act within the curve at some point of the region H, which leads to a point $\mathbf{P}^*$. Then the new plasticity angle at the point $\mathbf{P}^*$ differs from that constructed by the method of tangents from the point $\mathbf{P}^*$ to the original plasticity curve (for the point $\mathbf{P}$) in distance by a magnitude of the highest order of smallness in comparison with $|\Delta\mathbf{P}|$. In fact, in accordance with assumption B, the angle of plasticity at point $\mathbf{P}^*$ may not cross the original plasticity curve in the region H. Further, since the plasticity curve is convex and changes continuously, it follows that the sides of the plasticity angle at the point $\mathbf{P}^*$ may not get closer to the origin of coordinates than the tangents from the point $\mathbf{P}^*$ to the original curve, within an accuracy of deviations (in distance) of an order of magnitude higher as compared to $|\Delta\mathbf{P}|$. On the other hand, these sides cannot be further away from the origin of coordinates (within the same degree of accuracy) than the tangents indicated, because the latter (taking into account the non-concavity of the plasticity curve) correspond to the smallest displacement paths of the points of the plasticity curve, as is required by assumption C.

As is known, the condition

$$d\text{Э}^p = \Phi(p)\, d\mathbf{P} \qquad \left(\Phi(p) = \frac{d\text{Э}^p}{dp}\right) \tag{4.1}$$

is satisfied for simple loading from an initially isotropic state.

From the condition of symmetry it follows that the plasticity angle in this case is always symmetric with respect to the ray of loading. Let $p = p(\gamma)$, then

$$\Phi(p) = \Phi[p(\gamma)] = F(\gamma) \tag{4.2}$$

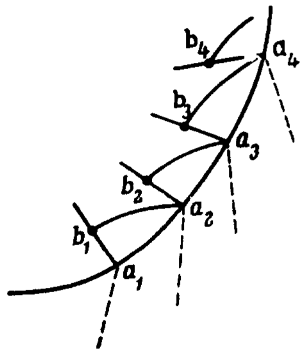

Fig. 1.

From assumption D it follows that if on some loading path (not necessarily a simple one) the vector of additional loading is directed at each point along the bisectrix of the running plasticity angle (bisectorial loading path) then for this path, as well as for the simple one, the relationship

$$d\text{Э}^{p} = F(\gamma_\sigma)\,d\mathbf{P} \tag{4.3}$$

holds good, where γ_σ is the magnitude of the plastic angle at the running point of the given bisectorial loading path.

5. Derivation of relations between the vectors of stress and plastic deformation. Let some loading path be given (Fig. 1). We replace it by a neighboring broken path constructed by the following rule. From some point a_1 of the given path we go along the sides of the plasticity angle at this point, directed to the same side of the bisectrix angle which is also tangent to the given path at the point a_1. Next, from the point b_1, close to a_1 and situated on the given side of the plasticity angle we go along the path b_1a_2 which possesses the property that its tangent at each point coincides with the bisectrix of the running angle of plasticity (bisectorial path). Further, it will be seen below that such a path will lead from b_1 to some point a_2 on the given path of loading. Next, from the point a_2 we go along the side of the plasticity angle at this point to the point b_2, and again by the bisectorial path from this point, etc.

We isolate a path element a_na_{n+1} and the corresponding broken path $a_nb_na_{n+1}$ (Fig. 2) and calculate the increase of plastic deformation along the broken path. Along the part a_nb_n, in accordance with assumption B, the increase of plastic deformation is equal to zero. On the basis of the results of the preceding section, relationship (4.3) is valid at each point of the path on the portion b_na_{n+1}, and the plasticity angle γ_σ is

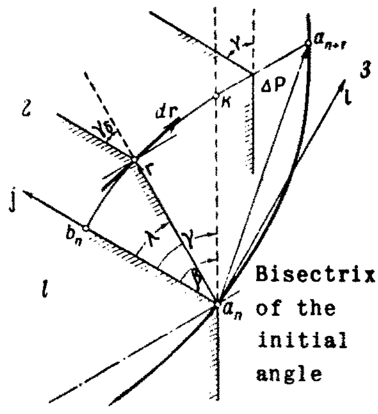

Fig. 2.

constructed as follows: the upper side of the plasticity angle at the running point of this path is parallel (with an accuracy indicated above) to the direction **j**, and the lower passes through the point a_n, if $0 \leqslant \lambda \leqslant \gamma$, and parallel to the direction ka_n, if $\lambda \geqslant \gamma$ (Fig. 2). Here γ is the initial plasticity angle at the point a_n, and λ is the angle formed by the running radius vector **r** of the bisectorial portion with the direction **j**. It is seen that $\gamma_\sigma = \lambda$ on the portion $0 \leqslant \lambda \leqslant \gamma$, while the bisectrix of the plasticity angle forms the angle $1/2\ \lambda$ with the perpendicular to **r** at each point. Therefore, the equation of bisectorial path on the portion from b_n to k is the equation

$$\frac{dr}{r a\lambda} = \operatorname{tg}\frac{\lambda}{2} \qquad \text{or} \qquad r = \frac{C}{\cos^2 {}^1\!/_2\lambda} \tag{5.1}$$

Let $\Delta \mathbf{P}$ designate the additional loading along the given path from the point a_n to the point a_{n+1} and let β be the angle formed by the direction of $\Delta \mathbf{P}$ with the direction **j**.

It is then not difficult to show that

$$C = \begin{cases} \Delta\sigma \cos^2 {}^1\!/_2\beta & \text{if } 0 \leqslant \beta \leqslant \gamma \\ \Delta\sigma \cos(\beta - {}^1\!/_2\gamma) \cos {}^1\!/_2\gamma & \text{if } \beta \geqslant \gamma \end{cases} \tag{5.2}$$

On the portion ka_{n+1} the bisectorial path is a straight line parallel to the bisectrix of the initial plasticity angle and $\gamma_\sigma = \gamma$. It is now obvious that the bisectorial path leads to the point on the given loading path.

The general increment of plastic deformation on the path $a_n b_n a_{n+1}$ is

$$\Delta\partial^p = \int_0^\beta d\partial_0^p = \int_0^\beta F(\gamma_6)\,d\mathbf{r} = \int_0^\beta F(\lambda)\,d\mathbf{r} = F(\lambda)\,\mathbf{r}\,|_0^\beta - \int_0^\beta \frac{dF(\lambda)}{d\lambda}\mathbf{r}\,d\lambda \tag{5.3}$$

Let us designate below by indeces 2 and 3 the zones $0 \leqslant \beta \leqslant \gamma$ and $\beta \geqslant \gamma$, and the elastic zone $\beta \leqslant 0$ by the index 1 (Fig. 2). (For example, $\Delta\partial_2{}^p$ is the increment of plastic deformation if the additional loading $\Delta\bar{\mathbf{P}}$ is derected toward the zone 2.) Inasmuch as $dF/d\lambda = 0$ for $\lambda \geqslant \gamma$ and inasmuch as the constant C takes on different values in zones 2 and 3, we obtain two different expressions for the increment of plastic deformation in the two zones. (We should bear in ming that $F(0) = 0$, $\mathbf{r}/_\beta = \Delta\mathbf{P}$.)

$$\Delta\partial_2{}^p = F(\beta)\,\Delta\mathbf{P} - \Delta\sigma\cos^2\frac{\beta}{2}\int_0^\beta \frac{dF(\lambda)}{d\lambda}\frac{\mathbf{n}}{\cos^2{}^1/_2\lambda}\,d\lambda$$

$$\Delta\partial_3{}^p = F(\gamma)\,\Delta\mathbf{P} - \Delta\sigma\cos\left(\beta - \frac{\gamma}{2}\right)\cos\frac{\gamma}{2}\int_0^\gamma \frac{dF(\lambda)}{d\lambda}\frac{\mathbf{n}}{\cos^2{}^1/_2\lambda}\,d\lambda \tag{5.4}$$

$$\mathbf{n} = \mathbf{i}\sin\lambda + \mathbf{j}\cos\lambda \tag{5.5}$$

For simple loading $\beta = 1/2\,\pi + 1/2\,\gamma$, $\Lambda = d$ and for the equation for the third zone, we find

$$d\partial^p = F(\gamma)\,d\mathbf{P} \tag{5.6}$$

Hence

$$F(\gamma) = \frac{d\partial^p}{dp} = \Phi(p) \tag{5.7}$$

The integrals on the right-hand sides of equations (5.4) are particularly simple to evaluate if we set

$$F(\gamma) = A(\gamma + \sin\gamma) \tag{5.8}$$

It is important to note that the function F selected in such a way provides the possibility of sufficiently well approximating the curves of uniaxial experiments. Taking F in the form (5.8), we integrate this relationship; taking into account that $\Delta\mathbf{P} = \mathbf{i}\sin\beta + \mathbf{j}\cos\beta$, as a result we obtain

$$\Delta\partial_2{}^p = A\Delta\sigma\,[\beta\sin\beta\,\mathbf{i} + (\rho\cos\rho - \sin\rho)\,\mathbf{j}]$$

$$\Delta\partial_3{}^p = A\Delta\sigma\,\{[\gamma\sin\beta + \sin\gamma\sin(\beta-\gamma)\,\mathbf{i} + [(\gamma\cos\beta - \sin\gamma\cos(\beta-\gamma)]\,\mathbf{j}\,\} \tag{5.9}$$

These equations, with an accuracy within small quantities of an order higher as compared to $\Delta\sigma$, give the increment of plastic deformation along the broken path. Proceeding to the limit as $\Delta\sigma$ is made an arbitrarily small $d\sigma$, in accordance with assumption A we obtain that along the initial loading path an infinitely small additional loading $\delta\mathbf{P}$ produces an infinitely small increment of plastic deformation $\delta\mathfrak{Э}^p$, to be evaluated by equation (5.9) in which the symbol Δ has to be replaced by δ.

Let us switch in the formulas obtained from reference vectors $\mathbf{i}$, $\mathbf{j}$ to reference vectors

$$\boldsymbol{\rho} = \frac{\mathbf{P}}{p}, \qquad \mathbf{q} = \frac{\delta\boldsymbol{\rho}}{|\delta\boldsymbol{\rho}|} \tag{5.10}$$

The position and magnitude of the plasticity angle with respect to these vectors will be characterized by angles ϕ and ψ (Fig. 3). We introduce the convention that the angle ϕ characterizes the deviation from the direction $\delta\rho$ of that side of the plasticity angle whose projection on this direction has a positive sense. The position of the vector $\delta\mathbf{P}$ in the new system of base vectors will be characterized by the angle α formed with the direction $\delta\boldsymbol{\rho}$. From Fig. 3 it is seen that

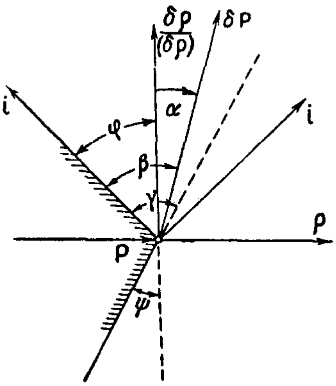

Fig. 3.

$$\mathbf{i} = \boldsymbol{\rho}\cos\varphi + \mathbf{q}\sin\varphi, \quad \mathbf{j} = -\boldsymbol{\rho}\sin\varphi + \mathbf{q}\cos\varphi \tag{5.11}$$

$$\gamma = \varphi + \psi, \qquad \alpha = \beta - \varphi, \qquad \sin\alpha = \frac{\delta p}{\delta\sigma}$$

We now evaluate the modulus of $\delta\boldsymbol{\rho}$. On the basis of the definition of the vector $\boldsymbol{\rho}$ we have

$$\delta\mathbf{P} = \delta(\boldsymbol{\rho}p) = \delta\boldsymbol{\rho}p + \delta p\boldsymbol{\rho}$$

Hence $\delta\sigma^2 = |\delta\boldsymbol{\rho}|^2 P^2 + (\delta p)^2$. Therefore,

$$|\delta\boldsymbol{\rho}| = \frac{1}{p}\sqrt{\delta\sigma^2 - \delta p^2} = \frac{1}{p}\delta\sigma\cos\alpha \tag{5.12}$$

On the basis of formulas (5.11), (5.12), the expressions (5.9), where we put $\Lambda = \delta$, are easily transformed to the form

$$\delta Э_2{}^p = A\left\{\delta p \rho\left[(\alpha + \varphi) + \frac{\sin(\alpha + \varphi)}{\sin\alpha}\sin\varphi\right] + p\delta\rho\left[(\alpha + \varphi) - \frac{\sin(\alpha + \varphi)}{\cos\alpha}\cos\varphi\right]\right\}$$

$$\delta Э_3{}^p = A\left\{\delta p \rho\left[(\varphi + \psi) + \frac{\sin(\varphi + \psi)}{\sin\alpha}\sin(\alpha + \varphi - \psi)\right] + p\delta\rho\left[(\varphi + \psi) - \frac{\sin(\varphi + \psi)}{\cos\alpha}\cos(\alpha + \varphi - \psi)\right]\right\} \tag{5.13}$$

As is easily seen, the components of the vector ρ are the components of the stress tensor $S_{ij}{}^* = S_{ijp}$, $(p^2 = \Sigma S_{ij}{}^2)$. Therefore, the tensorial form of relationships (5.13) is obtained by simply replacing $\delta Э^p$ by $\delta Э_{ij}{}^p$, ρ by $S_{ij}{}^*$, and $\sin\alpha$ by $\delta p \mid \delta\sigma$, $(\delta\sigma^2 = \Sigma\delta S_{ij}{}^2)$; ϕ and ψ are unknown scalar functions.

6. Certain supplements. Let an additional loading $\delta\mathbf{P}$ act from the state $\mathbf{P}$ which was reached by a simple loading. In this case we have to put in formulas (5.13) $\phi = \psi$, $\gamma = 2\gamma = \gamma(p)$. If the vector $\delta\mathbf{P}$ is in the zone 3 ($\alpha \geqslant 1/2\ \gamma = \phi$), then by formulas (5.13) we have

$$\delta Э_3^p = A\left[\delta p(\gamma + \sin\gamma)\rho + p(\gamma - \sin\gamma)\delta\rho\right] \tag{6.1}$$

Relationship (6.1) is reminiscent of the deformational law of Hencky-Nadai. For a full conformity it is required that

$$Ap(\gamma - \sin\gamma) = \int\Phi(p)\,\delta p = \chi(p) = Э^p \tag{6.2}$$

Then, taking into account (5.7), in fact we have

$$Э_s^p = \chi(p)\rho = (Э^p/_p)\rho$$

Let us clarify the meaning of condition (6.2). Differentiating with respect to p, we obtain

$$\frac{d}{dp}\left[p(\gamma - \sin\gamma)\right] = \gamma + \sin\gamma \tag{6.4}$$

Hence we find $p = c/\cos 1/2\,\gamma$. For $p = p_s$ we must have $\gamma = 0$, therefore

$$p = \frac{p_s}{\cos{}^1/_2\gamma} \tag{6.5}$$

Thus, for the relation (6.1) to coincide fully with the theory of Hencky-Nadai, the plastic angle in the process of simple loading should change in such a way that its sides always touch the initial plasticity circle $p = p_s$. (The latter is postulated in the theory of Sanders and follows from the theory of Batdorf-Budiansky.) There are grounds for assuming (experiments on stability of elasto-plastic structures) that the deformational theory should be valid within the limits of certain angles of additional loading. Within the framework of the assumptions laid down in the present paper this is possible for the zone 3 and therefore the change in the plasticity angle represented by equation (6.5) may possibly be close to the actual one.

We emphasize that the importance of the result obtained lies in the fact that it discloses the possibility of satisfying the deformational theory (within certain limits) without violating the basic assumptions of the mechanics of solid media (for example, the condition of continuity). This assumption coincides with the conclusion of the slip theory of Batdorf-Budiansky, even though the latter is based on entirely different assumptions from the present paper. We note that condition (6.5) of the present paper represents a particular case, and the theory admits, for example, of the introduction of an additional constant, which may improve the theory.

Even for condition (6.5) the relationships for the center zone remain very complicated, and we will not write them down.

If at the end of simple loading an additional loading occurs which has a component in the direction of $\delta\rho$, then the relationships obtained predict the appearance of plastic deformation in that direction. Let us determine the magnitude of the ratio of the component of plastic deformation in the direction $\delta\rho$ to the component of the additional loading in this direction $\delta q = \delta\sigma \cos a$.

From relationships (5.17), taking into account $|\delta\rho| = p^{-1} d\sigma \cos a$, we obtain

$$\frac{\delta Э_{\text{орт}}{}^p}{\delta q} = A \begin{cases} {}^1\!/_2 \left[(\gamma - \sin\gamma) + 2\alpha - 2\,\mathrm{tg}\,\alpha \cos^2 {}^1\!/_2\gamma\right] & (-{}^1\!/_2\gamma \leqslant \alpha \leqslant {}^1\!/_2\gamma) \\ (\gamma - \sin\gamma) & (\alpha \geqslant {}^1\!/_2\gamma) \end{cases} \qquad (6.6)$$

If condition (6.5) is assumed to be valid, then by formula (6.2) we have

$$A(\gamma - \sin\gamma) = \frac{\chi(p)}{p} = \frac{Э^p}{p}, \qquad \cos^2\frac{\gamma}{2} = \frac{p_s{}^2}{p^2} \qquad (6.7)$$

Hence

$$\frac{\delta Э_{\text{орт}}{}^p}{\delta q} = \begin{cases} \dfrac{1}{2}\dfrac{Э^p}{p} + \left(\alpha - \mathrm{tg}\,\alpha \dfrac{p_s{}^2}{p^2}\right), & \dfrac{1}{2}\gamma \leqslant \alpha \leqslant \dfrac{1}{2}\gamma \\ \dfrac{Э^p}{p} & \alpha \geqslant \dfrac{1}{2}\gamma \end{cases} \qquad (6.8)$$

If simple extension has taken place up to the point $\mathbf{P}$, then

$$Э^p = \sqrt{\Sigma Э_{ij}{}^p} = \sqrt{\frac{3}{2}}\,Э_x{}^p = \sqrt{\frac{3}{2}}\,\varepsilon_x{}^p$$

by virtue of elastic compressibility of material

$$p = \sqrt{\Sigma S_{ij}^2} = \sqrt{\frac{3}{2}}\,\sigma_x$$

Hence

$$\frac{\vartheta^p}{p} = \frac{3}{2}\frac{\varepsilon_x{}^p}{\sigma_x} = \frac{3}{2}\left(\frac{1}{E_s} - \frac{1}{E}\right), \qquad \frac{p_s}{p} = \frac{\sigma_{xs}}{\sigma_x} \tag{6.9}$$

where E_s is the shear modulus on the curve of simple extension, E is Young's modulus. Further, let the additional loading be given as a combination of extension $\delta\sigma_x$ and shear $\delta\tau_{xy}$. Thereby $/\delta q = \delta\vartheta^p_{xy}/\delta S_{xy} = (^1/_2)(\delta\gamma^p_{xy}/\delta\tau_{xy})$. The angle α will be determined by the relationship

$$\sin\alpha = \frac{\delta p}{\delta\sigma} = \frac{1}{p}\frac{\Sigma S_{ij}\delta S_{ij}}{(\Sigma\delta S_{ij}{}^2)^{1/2}} = \left(1 + 3\frac{\delta\tau_{xy}}{\delta\sigma_x}\right)^{-1/2} \tag{6.10}$$

and on the basis of what has been said above and (6.8) (the subscripts x, y are omitted) we have

$$\frac{\delta\gamma^p}{\delta\tau} = \begin{cases} \frac{3}{2}\left(\frac{1}{E_s} - \frac{1}{E}\right) + 2\left(\operatorname{arctg}\frac{1}{\sqrt{3}}\frac{\delta\sigma}{\delta\tau} - \frac{1}{\sqrt{3}}\frac{\delta\sigma}{\delta\tau}\right)\left(\frac{\sigma_s}{\sigma}\right)^2, & -\operatorname{tg}\frac{\gamma}{2} \leqslant \frac{1}{\sqrt{3}}\frac{\delta\sigma}{\delta\tau} \leqslant \operatorname{tg}\frac{\gamma}{2} \\ 3\left(\frac{1}{E_s} - \frac{1}{E}\right), & \frac{1}{\sqrt{3}}\frac{\delta\sigma}{\delta\tau} \geqslant \operatorname{tg}\frac{\gamma}{2} \end{cases}$$

The instantaneous plastic shear modulus G_i is determined by the formula

$$G_i = \frac{\delta\tau}{\delta\gamma} = \frac{\delta\tau}{\delta\gamma^p + \delta\gamma^y} - \frac{G}{1 + G\delta\gamma^p/\delta\tau} \tag{6.12}$$

and thus it is completely determined by the relationship (6.11). As is seen, for additional loading into zone 3, G_i does not depend on the direction of the additional loading $\delta\sigma/\delta\tau$ and coincides with that predicted by the deformational theory. For additional loading into zone 2, G_i changes smoothly with the change in direction of additional loading, from the value indicated above to $G_i = G$ on the boundary of the elastic region. For orthogonal additional loading $\delta\sigma = 0$ we have

$$\left[1 + \frac{3}{2}G\left(\frac{1}{E_s} - \frac{1}{E}\right)\right]G_i = G \tag{6.13}$$

These results coincide with the consequences of the slip theory of Batdorf-Budiansky.

7. Some qualitative deductions. It follows immediately from assumption D that for an additional loading from an arbitrary state **P** associated with the smooth point of the plastic curve, the same effects take place as for the passage beyond the initial plasticity circle, provided the orientation of additional loading with respect to the two differential elements is the same. In particular, it should be expected that for an initially extended specimen the shear curve in the vicinity of the yield point (possibly a changed one as compared to the initial one) may be of the form as the initial one.

As is seen from Section 6, the function $\gamma = \gamma(p)$ must be known in order to determine the increments of plastic deformation for an additional loading from the end of simple loading. In our calculations we determine this function by formula (6.5), and this was the only possibility of making the deformational theory valid not only for simple loading. Formula (6.5) permits of changing the plasticity angle in such a way that it can easily be generalized for the case of an arbitrary loading path in the form of the method of external tangents postulated in [4]. To provide a rigorous justification for such a method of construction of plastic angles does not appear possible within the framework of hypotheses which have been introduced; however, the application of the method of external tangents is reasonable for the verification of the theory and for the clarification of possibilities for its improvement.

The most essential conclusion of the theory employing the method of external tangents is the fact that for paths which deviate rather strongly from simple loading, a conformance with relationships of the theory of Hencky-Nadai is obtained. (The tangent vector to the path of such loading at an arbitrary point must lie within the angle formed by the tangents to the initial plasticity circle from the given point.) If, however, the tangent vector to the loading path at each point lies outside the indicated angle on one of its sides, then along the whole of such a path the relationships of the second zone are fulfilled, whereby the angle ϕ, entering the relationship, will be equal to the angle which would be obtained at this point for simple loading. A circular loading p = const, produced at the end of simple loading, falls into this class.

BIBLIOGRAPHY

1. Batdorf, I. and Budiansky, B., A Mathematical Theory of Plasticity Based on the Concept of Slip. *NACA*, TN No. 1871, April, 1949.

2. Lin, T.H., A Proposed Theory of Plasticity Based on Slips. *Proc. of the 2nd US Nat. Cogr. of Appl. Mech.*, pp. 461-468, 1954.

3. Koiter, W.T., Stress-Strain Relations, Uniqueness and Variational Theorems for Elastic-Plastic Materials with a Singular Yield Surface. *Quart. Appl. Math.* Vol. 11, No. 3, pp. 350-354, 1953.

4. Sanders, I.L., Plastic Stress-Strain Relations Based on Linear Loading Functions. *Proc. of the 2nd US Nat. Congr. of Appl. Mech.*, pp. 445-460, 1955.

5. Kliushnikov, V.D., O zakonakh plastichnosti dlia materiala s uprochneniem (On the plasticity laws for work-hardening materials). *PMM* Vol. 22, No. 1, 1958.

6. Kliushnikov, V.D., O zonakh plastichnosti dlia chastnogo klassa putei nagruzheniia (On the zones of plasticity for a particular class of loading paths). *PMM* Vol. 21, No. 4, 1957.

7. Drucker, D.C., A More Fundamental Approach to Plastic Stress-Strain Relations. *Proc. of the First US Nat. Congr. of Appl. Mech.*, ASME, 1951.

8. Naghdi, P.M., Essenburg, F. and Koff, W., An Experimental Study of Initial and Subsequent Yield Surfaces in Plasticity. *J. Appl. Mech.*, Vol. 25, No. 2, 1958.

Translated by M.M.S.